

# $\mathcal{O}(\alpha_s v^2)$ correction to $e^+e^- \rightarrow J/\psi + \eta_c$ at $B$ factories

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We investigate the  $\mathcal{O}(\alpha_s v^2)$  correction to the  $e^+e^- \rightarrow J/\psi + \eta_c$  process in the nonrelativistic QCD (NRQCD) factorization approach. Within some reasonable choices of the relative order- $v^2$  NRQCD matrix elements, we find that including this new ingredient of correction only mildly enhances the existing NRQCD predictions. We have also deduced the asymptotic expressions for the  $\mathcal{O}(\alpha_s v^2)$  short-distance coefficients, and reconfirm the early speculation that at next-to-leading order in  $\alpha_s$ , the double logarithm of type  $\ln^2(s/m_c^2)$  appearing in various NRQCD short-distance coefficients is always associated with the helicity-suppressed channels.

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## I. INTRODUCTION

One of the most intensively studied hard exclusive reactions in recent years is perhaps the double-charmonium production process  $e^+e^- \rightarrow J/\psi + \eta_c$  at the  $B$  factory energy  $\sqrt{s} = 10.58$  GeV [1]. It was initially measured by the BELLE experiment in 2002 with  $\sigma[e^+e^- \rightarrow J/\psi + \eta_c] \times \mathcal{B}_{\geq 4} = 33_{-6}^{+7} \pm 9$  fb [2], where  $\mathcal{B}_{\geq 4}$  is the branching ratio of  $\eta_c$  into 4 or more charged tracks. The first theoretical predictions [3–5], built on the lowest-order (LO) calculation in the nonrelativistic QCD (NRQCD) factorization approach [6], were scattered in the range 2.3–5.5 fb, almost one order of magnitude smaller than the BELLE data. Later BELLE Collaboration refined their measurement and gave  $\sigma[J/\psi + \eta_c] \times \mathcal{B}_{>2} = 25.6 \pm 2.8 \pm 3.4$  fb [7], where  $\mathcal{B}_{>2}$  denotes the branching fraction for the  $\eta_c$  into more than two charged tracks. In 2005, BABAR Collaboration also measured the same observable and obtained  $17.6 \pm 2.8_{-2.1}^{+1.5}$  fb [8].

The disquieting discrepancy between experiment and the LO NRQCD predictions has spurred a great amount of theoretical endeavors in the following years. Roughly speaking, most works can be divided into two major categories, either based on the light-cone factorization [9–12], or based on the NRQCD factorization approach [13–16] (for the investigations from other theoretical approaches, see Refs. [17–20]).

One crucial step toward alleviating the tension between data and theory is the discovery of the positive and significant  $\mathcal{O}(\alpha_s)$  correction to the  $e^+e^- \rightarrow J/\psi + \eta_c$  pro-

cess [13, 14]. This next-to-leading order (NLO) perturbative calculation was performed in the NRQCD factorization framework (a similarly large positive  $\mathcal{O}(\alpha_s)$  correction has also later been found for the  $e^+e^- \rightarrow J/\psi + \chi_{c0}$  process [21, 22]). In contrast, due to some long-standing theoretical difficulty inherent to the helicity-suppressed process, by far no one has successfully conducted the corresponding  $\mathcal{O}(\alpha_s)$  correction to this process in the light-cone approach. In a sense, for the  $e^+e^- \rightarrow J/\psi + \eta_c$  process, NRQCD approach seems more systematic and maneuverable than the light-cone approach.

The relative  $\mathcal{O}(v^2)$  correction to  $e^+e^- \rightarrow J/\psi + \eta_c$  has also been addressed [3, 15, 16], where  $v$  denotes the typical velocity of the  $c$  quark in a charmonium. Notwithstanding the large uncertainty inherent to relativistic correction, it was believed that [15, 16], including both NLO perturbative and (a partial resummation of ) relativistic corrections, one may achieve the reasonable agreement, albeit with large uncertainties, between the NRQCD prediction and  $B$  factory data.

The goal of this work is to address the  $\mathcal{O}(\alpha_s v^2)$  correction to  $e^+e^- \rightarrow J/\psi + \eta_c$  in the NRQCD factorization approach. It is curious to examine its phenomenological impact. On the other hand, thus far there are only very few basic quarkonium decay processes whose  $\mathcal{O}(\alpha_s v^2)$  corrections have been calculated, *e.g.*  $J/\psi \rightarrow e^+e^-$  [23], and  $\eta_c \rightarrow \gamma\gamma$  [24, 25]. Therefore, it is also theoretically interesting to know the  $\mathcal{O}(\alpha_s v^2)$  effect for the exclusive quarkonium production process in the first time.

The rest of the paper is structured as follows. In Secs. II and III, we express the product rate for  $e^+e^- \rightarrow J/\psi + \eta_c$  in terms of the  $J/\psi + \eta_c$  electromagnetic (EM) form factor, and present the NRQCD factorization formulas for both quantities, accurate through relative order- $v^2$ . In Sec. IV, we list the tree-level short-distance coefficients through relative  $\mathcal{O}(v^2)$ . In Sec. V, we first

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sketch some key technical steps about the NLO perturbative calculations, then present the asymptotic expressions for the matching coefficients. We devote Sec. VI to exploring the phenomenological impact of our new  $\mathcal{O}(\alpha_s v^2)$  correction on the  $B$  factory measurement. Finally, we summarize in Sec. VII.

## II. $J/\psi + \eta_c$ EM FORM FACTOR

Suppose we work in the  $e^-$  and  $e^+$  center-of-mass frame with invariant mass of  $\sqrt{s}$ . Let  $P_1$  ( $\lambda$ ) denote the momentum (helicity) of the  $J/\psi$ , and  $P_2$  the momentum of the  $\eta_c$ , respectively. This process simply probes the  $J/\psi + \eta_c$  electromagnetic form factor in the timelike region, referred to as  $G(s)$  hereafter:

$$\langle J/\psi(P_1, \lambda) + \eta_c(P_2) | J_{\text{EM}}^\mu | 0 \rangle = i G(s) \epsilon^{\mu\nu\rho\sigma} P_{1\nu} P_{2\rho} \varepsilon_\sigma^*(\lambda), \quad (1)$$

where  $J_{\text{EM}}^\mu$  is the electromagnetic current. The tensor structure specified in (1) is uniquely dictated by the Lorentz and parity invariance. As a result, the outgoing  $J/\psi$  must be transversely polarized, i.e.,  $\lambda = \pm 1$ .

The cross section can be expressed as

$$\sigma[e^+ e^- \rightarrow J/\psi + \eta_c] = \frac{4\pi\alpha^2}{3} \left( \frac{|\mathbf{P}|}{\sqrt{s}} \right)^3 |G(s)|^2, \quad (2)$$

where  $|\mathbf{P}|$  signifies the magnitude of the 3-momentum carried by the  $J/\psi$  ( $\eta_c$ ) in the center-of-mass frame. The cubic power of  $|\mathbf{P}|$  is reminiscent of the fact that  $J/\psi$  and  $\eta_c$  are in the relative  $P$ -wave orbital state.

It is worth recalling the asymptotic behavior of (2) for this helicity-flipped process. The helicity selection rule [26] dictates that  $G(s) \sim 1/s^2$  as  $\sqrt{s} \gg m_c$  [10], hence  $\sigma[J/\psi + \eta_c] \sim 1/s^4$ . Evidently, the charm quark mass may serve as the agent of violating the hadron helicity conservation. As will be examined in Sec. V, this power-law scaling is subject to double-logarithmic modification once beyond LO in  $\alpha_s$ .

## III. NRQCD FACTORIZATION FORMULA FOR $J/\psi + \eta_c$ PRODUCTION RATE

According to NRQCD factorization formula, the  $J/\psi + \eta_c$  EM form factor in (1) can be factorized as

$$G(s) = \sqrt{4M_{J/\psi} M_{\eta_c}} \langle J/\psi | \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi | 0 \rangle \langle \eta_c | \psi^\dagger \chi | 0 \rangle [c_0 + c_{2,1} \langle v^2 \rangle_{J/\psi} + c_{2,2} \langle v^2 \rangle_{\eta_c} + \dots], \quad (3)$$

where  $c_0$  and  $c_2$  are the corresponding short-distance coefficients. We have adopted relativistic normalization for the quarkonia states appearing in the left side, while using the nonrelativistic normalization for those in the NRQCD matrix elements. For simplicity, we have introduced the following dimensionless ratios of NRQCD

matrix elements to characterize the  $\mathcal{O}(v^2)$  corrections:

$$\langle v^2 \rangle_{J/\psi} = \frac{\langle J/\psi(\lambda) | \psi^\dagger (-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2 \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}(\lambda) \chi | 0 \rangle}{m_c^2 \langle J/\psi(\lambda) | \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}(\lambda) \chi | 0 \rangle}, \quad (4a)$$

$$\langle v^2 \rangle_{\eta_c} = \frac{\langle \eta_c | \psi^\dagger (-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2 \chi | 0 \rangle}{m_c^2 \langle \eta_c | \psi^\dagger \chi | 0 \rangle}, \quad (4b)$$

where  $\psi^\dagger \overleftrightarrow{\mathbf{D}} \chi \equiv \psi^\dagger \mathbf{D} \chi - (\mathbf{D} \psi)^\dagger \chi$ .

Substituting Eq. (3) into (2), one can decompose the cross section into the  $\mathcal{O}(v^0)$  and  $\mathcal{O}(v^2)$  pieces:

$$\sigma[e^+ e^- \rightarrow J/\psi + \eta_c] = \sigma_0 + \sigma_2 + \mathcal{O}(\sigma_0 v^4), \quad (5)$$

where

$$\sigma_0 = \frac{8\pi\alpha^2 m_c^2 (1-4r)^{3/2}}{3} \langle \mathcal{O}_1 \rangle_{J/\psi} \langle \mathcal{O}_1 \rangle_{\eta_c} |c_0|^2, \quad (6a)$$

$$\sigma_2 = \frac{4\pi\alpha^2 m_c^2 (1-4r)^{3/2}}{3} \langle \mathcal{O}_1 \rangle_{J/\psi} \langle \mathcal{O}_1 \rangle_{\eta_c} \left\{ \left( \frac{1-10r}{1-4r} |c_0|^2 + 4 \text{Re}[c_0 c_{2,1}^*] \right) \langle v^2 \rangle_{J/\psi} + \left( \frac{1-10r}{1-4r} |c_0|^2 + 4 \text{Re}[c_0 c_{2,2}^*] \right) \langle v^2 \rangle_{\eta_c} \right\}. \quad (6b)$$

In deriving (6), we have employed the Gremm-Kapustin relation [27]  $M_H^2 \approx 4m_c^2(1 + \langle v^2 \rangle_H)$  to eliminate the explicit occurrences of  $M_{J/\psi}$  and  $M_{\eta_c}$ . For notational brevity, we have introduced the following symbols:  $r = \frac{4m_c^2}{s}$ ,  $\langle \mathcal{O}_1 \rangle_{J/\psi} = |\langle J/\psi | \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi | 0 \rangle|^2$ , and  $\langle \mathcal{O}_1 \rangle_{\eta_c} = |\langle \eta_c | \psi^\dagger \chi | 0 \rangle|^2$ .

It is convenient to organize the short-distance coefficients  $c_i$  in power series of the strong coupling constant, i.e.,  $c_i = c_i^{(0)} + \frac{\alpha_s}{\pi} c_i^{(1)} + \dots$ . Accordingly, we may decompose the cross section  $\sigma_i$  into  $\sigma_i^{(0)} + \sigma_i^{(1)}$  ( $i = 0, 2$ ). Our primary goal in this work is to calculate  $\sigma_2^{(1)}$ .

## IV. TREE-LEVEL NRQCD SHORT-DISTANCE COEFFICIENTS

It is straightforward to employ the perturbative matching method to determine the short-distance coefficients, by replacing the  $J/\psi$  and  $\eta_c$  states with the free  $c\bar{c}(^3S_1^{(1)})$  and  $c\bar{c}(^1S_0^{(1)})$  pairs, and enforcing that both perturbative QCD and NRQCD calculations in (3) yield the same answer, order by order in  $\alpha_s$ .

There are in total 4 diagrams at LO in  $\alpha_s$ , one of which is depicted in Fig. 1(a). The tree-level short-distance coefficients through  $\mathcal{O}(v^2)$  have been available long ago [3]. Here we list their values at  $D = 4 - 2\epsilon$  spacetime dimen-

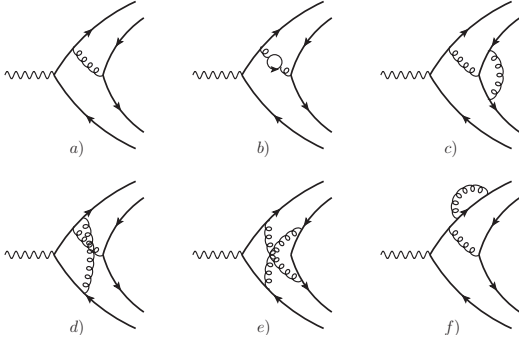


FIG. 1: One sample LO diagram and five sample NLO diagrams that contribute to  $\gamma^* \rightarrow J/\psi + \eta_c$ .

sion:

$$c_0^{(0)} = \frac{32\pi C_F e_c \alpha_s}{N_c m_c s^2}, \quad (7a)$$

$$c_{2,1}^{(0)} = c_0^{(0)} \left[ \frac{3-10r}{6} + \left(1 - \frac{16}{9}r\right) \epsilon + \mathcal{O}(\epsilon^2) \right], \quad (7b)$$

$$c_{2,2}^{(0)} = c_0^{(0)} \left[ \frac{2-5r}{3} + \left(\frac{10}{9} - \frac{16}{9}r\right) \epsilon + \mathcal{O}(\epsilon^2) \right], \quad (7c)$$

where  $e_c = \frac{2}{3}$  is the electric charge of the charm quark,  $N_c = 3$  is the number of colors and  $C_F = \frac{N_c^2 - 1}{2N_c}$ .

## V. NLO PERTURBATIVE NRQCD SHORT-DISTANCE COEFFICIENTS

In this section, we first sketch some technical issues about the NLO perturbative calculations, then present the analytic expressions of the NLO corrections to the matching coefficients  $c_0$  and  $c_2$ . Technical details will be expounded in a long write-up [28].

### A. Description of the calculation

The main technical difficulty is to identify the relativistic effects in the quark amplitude  $\gamma^* \rightarrow c\bar{c}(P_1, {}^3S_1^{(1)}) + c\bar{c}(P_2, {}^1S_0^{(1)})$  through NLO in  $\alpha_s$ . The momenta of  $c$  and  $\bar{c}$  inside each pair are assigned as  $p_i = \frac{1}{2}P_i + q_i$ ,  $\bar{p}_i = \frac{1}{2}P_i - q_i$  for  $i = 1, 2$ . The total momentum  $P_i$  and relative momentum  $q_i$  are chosen to be orthogonal. In the rest frame of each  $c\bar{c}$  pair, the total and relative 4-momenta read  $P_i^\mu = (2E_{q_i}, \mathbf{0})$ ,  $q_i^\mu = (0, \mathbf{q}_i)$ , respectively, where  $E_{q_i} = \sqrt{m_c^2 + \mathbf{q}_i^2}$ .

We first employ the MATHEMATICA package FEYNARTS [29, 30] to generate Feynman diagrams and amplitudes for the process  $\gamma^* \rightarrow c(p_1)\bar{c}(\bar{p}_1) + c(p_2)\bar{c}(\bar{p}_2)$  to NLO in  $\alpha_s$ . In total there are 20 two-point, 20 three-point, 18 four-point, and 6 five-point one-loop diagrams, some of which have been illustrated in Fig. 1(b)-(f). We

then apply the covariant spin projector [31] to enforce two  $c\bar{c}$  pairs to form the spin-triplet/singlet, color-singlet states, with the Dirac and color traces handled by FEYN-CALC [32].

We then expand the amplitude in powers of the quark relative momenta,  $q_i$ , up to the quadratic order. We then make the following substitution to project out the  $S$ -wave states:

$$q_i^\mu q_i^\nu \rightarrow \frac{\mathbf{q}_i^2}{D-1} \left( -g^{\mu\nu} + \frac{P_i^\mu P_i^\nu}{P_i^2} \right), \quad (8)$$

for  $i = 1, 2$ , and  $\mathbf{q}_i^2$  is understood to be defined in the rest frame of each  $c\bar{c}$  pair.

In conventional matching procedure, one expands the relative momentum  $q_i$  only after completing the loop integrations in the QCD amplitude, which is a daunting task in our case since the entanglement of three disparate scales,  $\sqrt{s}$ ,  $m_c$ ,  $\mathbf{q}_i$ , in a loop integral. In this work, we employ a much simpler shortcut suggested by the *method of region* [33], i.e., making expansion in  $q_i$  prior to carrying out the loop integration. This amounts to directly extracting the NRQCD short-distance coefficients, i.e., the contributions solely arising from the *hard* region ( $k^2 \geq m_c^2$ ). Consequently, we will no longer be distracted by the effects from the low-energy regions such as the *potential* ( $k^0 \sim mv^2$ ,  $|\mathbf{k}| \sim mv$ ) region.

To proceed, we use the MATHEMATICA packages FIRE [34] and APART [35] to reduce the general higher-point one-loop tensor integrals into a set of master integrals. As a bonus of having expanded the integrand in powers of  $q_i$ , and utilized a trick of rescaling the pair momentum  $P_i$  to make some hidden relativistic effects explicit, it turns out that all the required master integrals are just the usual 1-, 2- and 3-point scalar integrals, all of which can be found in the Appendix of Ref. [14].

We adopt the dimensional regularization to regularize both UV and IR singularities, with spacetime dimension  $D = 4 - 2\epsilon$ . We use 't Hooft-Veltman scheme to handle  $\gamma_5$  [36, 37]. After summing the contributions from all the diagrams, and incorporating mass and coupling constant renormalization, we find that the ultimate NLO expressions for the  $\mathcal{O}(v^0)$  QCD amplitude are both UV and IR finite, while the  $\mathcal{O}(v^2)$  amplitude is UV finite albeit IR divergent.

The occurrence of the IR divergences in the hard region at  $\mathcal{O}(\alpha_s v^2)$  is just as expected. From the pull-up mechanism, one can identify the IR divergences encountered in the hard-region calculation with those would arise from the *soft* ( $k^\mu \sim mv$ ) region in a literal QCD-side calculation, which must be canceled out upon matching. In fact, these IR divergences can be reconstructed with the knowledge of the  $\mathcal{O}(\alpha_s v^2)$  correction to the perturbative NRQCD matrix elements  $\langle c\bar{c}({}^3S_1) | \psi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \chi | 0 \rangle^{(1)}$  [23],  $\langle c\bar{c}({}^1S_0) | \psi^\dagger \chi | 0 \rangle^{(1)}$  [24], as well as  $c_0^{(0)}$  [28]. Thus, we finally end up with the both UV, IR-finite short-distance coefficients  $c_i^{(1)}$  ( $i = 0, 2$ ). Our finding is consistent with the all-order-in- $\alpha_s$  proof outlined in [38], that NRQCD

factorization holds for the exclusive production of one  $S$ -wave quarkonium plus any higher orbital angular momentum quarkonium in  $e^+e^-$  annihilation.

### B. Asymptotic expressions of NLO short-distance coefficients

The NRQCD short-distance coefficients  $c_0^{(1)}$  and  $c_2^{(1)}$  are in general complex-valued. Their analytic expressions

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$$c_0^{(1)}\left(r, \frac{\mu_r^2}{s}\right)_{\text{asym}} = c_0^{(0)} \times \left\{ \beta_0 \left( -\frac{1}{4} \ln \frac{s}{4\mu_r^2} + \frac{5}{12} \right) + \left( \frac{13}{24} \ln^2 r + \frac{5}{4} \ln 2 \ln r - \frac{41}{24} \ln r - \frac{53}{24} \ln^2 2 + \frac{65}{8} \ln 2 - \frac{1}{36} \pi^2 - \frac{19}{4} \right) + i\pi \left( \frac{1}{4} \beta_0 + \frac{13}{12} \ln r + \frac{5}{4} \ln 2 - \frac{41}{24} \right) \right\}, \quad (9a)$$

$$c_{2,1}^{(1)}\left(r, \frac{\mu_r^2}{s}, \frac{\mu_f^2}{m_c^2}\right)_{\text{asym}} = \frac{1}{2} c_0^{(0)} \times \left\{ \frac{16}{9} \ln \frac{\mu_f^2}{m_c^2} + \beta_0 \left( -\frac{1}{4} \ln \frac{s}{4\mu_r^2} + \frac{11}{12} \right) + \left( \frac{3}{8} \ln^2 r + \frac{19}{12} \ln 2 \ln r + \frac{31}{24} \ln r - \frac{1}{24} \ln^2 2 + \frac{893}{216} \ln 2 - \frac{5}{36} \pi^2 - \frac{497}{72} \right) + i\pi \left( \frac{1}{4} \beta_0 + \frac{3}{4} \ln r + \frac{19}{12} \ln 2 + \frac{9}{8} \right) \right\}, \quad (9b)$$

$$c_{2,2}^{(1)}\left(r, \frac{\mu_r^2}{s}, \frac{\mu_f^2}{m_c^2}\right)_{\text{asym}} = \frac{2}{3} c_0^{(0)} \times \left\{ \frac{4}{3} \ln \frac{\mu_f^2}{m_c^2} + \beta_0 \left( -\frac{1}{4} \ln \frac{s}{4\mu_r^2} + \frac{2}{3} \right) + \left( \frac{1}{12} \ln^2 r + \frac{11}{12} \ln 2 \ln r - \frac{1}{24} \ln r - \frac{11}{8} \ln^2 2 + \frac{241}{144} \ln 2 - \frac{1}{8} \pi^2 - \frac{99}{16} \right) + i\pi \left( \frac{1}{4} \beta_0 + \frac{1}{6} \ln r + \frac{11}{12} \ln 2 - \frac{1}{24} \right) \right\}, \quad (9c)$$


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where  $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$  is the one-loop coefficient of the QCD  $\beta$  function, and  $n_f = 4$  denotes the number of active quark flavors.  $\mu_r$  denotes the renormalization scale, with the natural magnitude of order  $\sqrt{s}$ ; and  $\mu_f$  is identified with the factorization scale in the  $\overline{\text{MS}}$  scheme, whose value lies in somewhere between  $m_c v$  and  $m_c$ , the UV cutoff scale of NRQCD.

It has been pointed out [39] that a peculiar double-logarithmic correction  $\propto \ln^2 r$  arises in  $c_0^{(1)}$  for this helicity-flipped process, and our (9a) exactly agrees with the corresponding expression there. Equations (9) imply that the presence of the double logarithm persists to  $\mathcal{O}(v^2)$  as well. Presumably, provided that  $\sqrt{s} \gg m_c$ , one must resum these types of large logarithms to all orders to obtain a reliable prediction. Unfortunately, due to our incapability for the light-cone approach to dealing with the helicity-suppressed hard exclusive processes beyond LO in  $\alpha_s$ , currently it remains to be an open question how to fulfill such a resummation.

are somewhat lengthy and will not be reproduced here. Nevertheless, it is enlightening to know their asymptotic expressions in the limit  $\sqrt{s} \gg m_c$ :

## VI. PHENOMENOLOGY

In the numerical analysis, we take  $\sqrt{s} = 10.58$  GeV, and the QED coupling constant  $\alpha(\sqrt{s}) = 1/130.9$  [16]. The running strong coupling constant is evaluated by using the two-loop formula with  $\Lambda_{\overline{\text{MS}}}^{(4)} = 0.338$  GeV [13, 14]. The LO NRQCD matrix elements are taken from [22]:  $\langle \mathcal{O}_1 \rangle_{J/\psi} \approx \langle \mathcal{O}_1 \rangle_{\eta_c} = 0.387 \text{ GeV}^3$ . If we choose  $m_c = 1.4$  GeV ( $r = 0.0700$ ), the Gremm-Kapustin relation implies that  $\langle v^2 \rangle_{J/\psi} = 0.223$  and  $\langle v^2 \rangle_{\eta_c} = 0.133$ . We will take  $\mu_f = m_c$ .

We are ready to carry out a detailed analysis for the processes  $e^+e^- \rightarrow J/\psi + \eta_c$  and confront the  $B$  factory measurements. One important source of theoretical uncertainties comes from the scale setting for the strong coupling constant. There is no way to circumvent the scale ambiguity problem within the confine of NRQCD factorization, and we proceed to estimate the cross section by affiliating all the occurring  $\alpha_s$  with a common scale,  $\mu_r$ , and choosing  $\mu_r = \sqrt{s}/2$  and  $\mu_r = 2m_c$ , respectively. It is hoped that the less biased results interpolate between these two sets of predictions.

With  $r = 0.07$ , for  $\mu_r = \sqrt{s}/2$ , we find  $c_0^{(1)}/c_0^{(0)} =$

TABLE I: Individual contributions to the predicted  $\sigma[e^+e^- \rightarrow J/\psi + \eta_c]$  at  $\sqrt{s} = 10.58$  GeV, labeled by powers of  $\alpha_s$  and  $v$ . The cross sections are in units of fb.

$\alpha_s(\mu_r)$	$\sigma_0^{(0)}$	$\sigma_0^{(1)}$	$\sigma_2^{(0)}$	$\sigma_2^{(1)}$
$\alpha_s(\frac{\sqrt{s}}{2}) = 0.211$	4.40	5.22	1.72	0.73
$\alpha_s(2m_c) = 0.267$	7.00	7.34	2.73	0.24

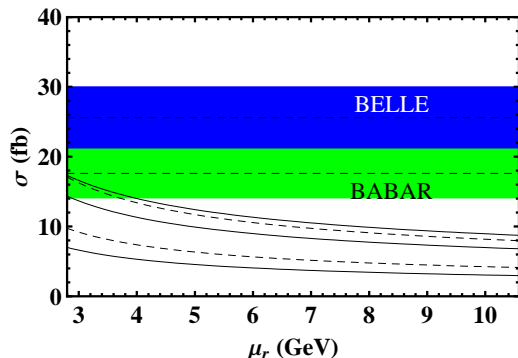


FIG. 2: The  $\mu_r$  dependence of the cross section for  $e^+e^- \rightarrow J/\psi + \eta_c$  at  $\sqrt{s} = 10.58$  GeV. The 5 curves from bottom to top are  $\sigma_0^{(0)}$  (solid line),  $\sigma_0^{(0)} + \sigma_2^{(0)}$  (dashed line),  $\sigma_0^{(0)} + \sigma_0^{(1)}$  (solid line),  $\sigma_0^{(0)} + \sigma_2^{(0)} + \sigma_0^{(1)}$  (dashed line), and  $\sigma_0^{(0)} + \sigma_2^{(0)} + \sigma_0^{(1)} + \sigma_2^{(1)}$  (solid line), respectively. The blue and green bands represent the measured cross sections by the BELLE and BABAR experiments, with respective systematic and statistical errors added in quadrature.

$8.83 - 6.02i$ ,  $c_{2,1}^{(1)}/c_{2,1}^{(0)} = -5.09 + 8.89i$ , and  $c_{2,2}^{(1)}/c_{2,2}^{(0)} = -5.15 + 8.99i$ ; for  $\mu_r = 2m_c$  we have  $c_0^{(1)}/c_0^{(0)} = 6.18 - 6.02i$ ,  $c_{2,1}^{(1)}/c_{2,1}^{(0)} = -7.74 + 8.89i$ , and  $c_{2,2}^{(1)}/c_{2,2}^{(0)} = -7.80 + 8.99i$ . Since  $\text{Re}[c_{2,i}^{(1)}/c_{2,i}^{(0)}]$  is large and negative for  $i = 1, 2$ , in conjunction with (7), one may think that the new  $\mathcal{O}(\alpha_s v^2)$  correction would dilute the known  $\mathcal{O}(v^2)$  effect.

Table I lists the predicted cross sections for  $e^+e^- \rightarrow J/\psi + \eta_c$  in double expansions of  $\alpha_s$  and  $v$ , with two sets of  $\mu_r$ . We reproduce the well-known results, i.e., the positive and significant  $\mathcal{O}(\alpha_s)$  correction [13, 14], and the positive but less pronounced  $\mathcal{O}(v^2)$  correction [3, 15, 16].

The new  $\mathcal{O}(\alpha_s v^2)$  ingredient,  $\sigma_2^{(1)}$ , is positive but modest. It may be attributed to the *near cancellation* between the two terms in the prefactor of  $\langle v^2 \rangle_H$  in (6b).

In Fig. 2, we plot the  $\sigma[e^+e^- \rightarrow J/\psi + \eta_c]$  as a function of  $\mu_r$ . Whether including the contribution of  $\sigma_2^{(1)}$  or not clearly does not make a big difference. When  $\mu_r$  is relatively small, the state-of-the-art NRQCD prediction converges to the BABAR measurement within errors. Had we taken somewhat larger values of the NRQCD matrix elements  $\langle O_1 \rangle_H$  as in [15, 16], the agreement with the two  $B$  factories measurements would be better.

## VII. SUMMARY

In this work we have computed the  $\mathcal{O}(\alpha_s v^2)$  correction to the helicity-suppressed process  $e^+e^- \rightarrow J/\psi + \eta_c$  in the NRQCD factorization framework. The corresponding NLO perturbative short-distance coefficients associated with the  $J/\psi + \eta_c$  EM form factor are directly extracted from the *hard* loop-momentum region through relative order  $v^2$ . By examining the asymptotic form of these coefficients, we reconfirm the pattern recognized in [22, 39]: The hard exclusive processes involving double-charmonium at higher twist in general are plagued with double logarithms of form  $\ln^2 s/m_c^2$  once beyond LO in  $\alpha_s$ . When  $\sqrt{s} \gg m_c$ , in order to obtain the reliable predictions for such types of processes, one is enforced to resum these potentially large double logarithms to all orders in  $\alpha_s$ , which so far remains to be an open challenge.

At the  $B$  factory energy, we found that incorporating this new piece of correction only modestly enhances the existing NRQCD predictions. This may be ascribed to some accidental cancellation between two different sources of relativistic corrections. At much higher  $\sqrt{s}$ , this  $\mathcal{O}(\alpha_s v^2)$  correction would be much more relevant.

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